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THE MATHEMATICS TEACHER

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"ELEMENTARY GEOMETRY" AND THE "FOUNDATIONS"

By H. E. WEBB
Central High School, Newark, N. J.

Recent attempts at reform or improvement in the teaching of elementary geometry have been directed chiefly toward the development of a preliminary informal treatment of the material with which the subject is concerned, or toward a judicious selection of "real problems" for the purpose of applying geometrical principles. The field of the abstract "original exercise" has long since been cultivated to the point of exhaustion.

But in spite of these efforts, the body of doctrine called "Plane Geometry" has remained in much the same form as in earlier epochs; and those interested in the theory of education continue to value the subject chiefly as a training in logic. The difficulty with this viewpoint, as it presents itself to the teacher, lies in the fact that as text books in geometry are written at present, the logic is at the same time so imperfect that brighter students are able to find flaws in it, and so abstruse that many students fail to grasp its meaning at all.

The effort to correct errors in the logic of Euclid, who in point of logic is superior to ninety-nine per cent of his successors in authorship, led Hilbert, and others after him, to enunciate a new beginning for the subject, which should not fall foul of Euclid's difficulties with undefined elements and tacit assumptions. It is not the province of this paper to discuss the relative merits of various sets of assumptions which have been proposed, beyond the statement that a choice among them is most likely governed by aesthetic considerations, and that they are all too abstract for the purposes of the beginner.

But a logical treatment of geometry must begin somewhere, or be dropped altogether. Unfortunately, in the effort to imitate Euclid and at the same time to make the subject easier, there has been apparent in the writing of texts a tendency to insist upon meticulous nicety of *form* coupled with gross negligence of the *logical content* of demonstrations. It is the belief

of the writer that the logical sense of the student of elementary geometry is often much keener than is generally assumed to be the case, and that if "demonstrative geometry" is to be included at all in the list of high-school subjects, the significance of logical deduction should nowhere be lost sight of. If it is granted, however, as it must be, that a purely formal geometric science is beyond the powers of a student of high-school age, it is clear that there should be a distinct line of separation between "Foundations of Geometry," as an exercise in formal logic, and "Elementary Geometry," as a study of space relations, which takes the conclusions of the former as a point of departure. The establishing of such a distinction is an arbitrary business, of course. But there are a number of eminent mathematicians in this country who are qualified to consider this question, and whose conclusions would doubtless be acceptable to all interested.

The preparation of a list of fundamental principles for "Elementary Geometry," in this sense, would make it possible for secondary teachers to present the subject in its proper light as one of several alternative sciences which may conceivably underlie celestial and terrestrial mechanics. There would no longer be any pretense of reducing an elementary science to a set of divinely inspired "axioms." Emphasis would be placed, properly, upon the necessity of making space relations explicit rather than intricate. And finally, it would be possible to eliminate from the subject certain persistent errors which have been copied from Euclid, or which have arisen in the effort to simplify Euclid without impairing his logic.

As such a list would make no pretense to mutual independence of its elements, it would necessitate absolutely the abandoning of the notion that, in the teaching of geometry, everything should be proved that can conceivably be proved. It would, however, adhere as closely as any mathematical philosophy to the ideal of consistency, and would provide carefully against tacit assumptions and inadequate definitions.

It is true that the National Committee on Mathematical Requirements has prefaced its list of propositions with a list of assumptions which accords in a measure with the foregoing plan. The committee is careful to designate its choice of principles as typical rather than conclusive. The publication of this list was

a long move in the right direction. It was promptly checked, however, by the College Entrance Examination Board in its recent report, which has included one assumption from this list in *its* list of starred propositions, and one or two others in its list of theorems requiring proof, but has substituted no list of fundamentals of its own.

Both lists, however, include as requiring proof the theorem that two triangles are congruent if two sides and the included angle of one are equal respectively to two sides and the included angle of the other, an action which to the uninitiated would seem to treat the judgment of such an authority as Hilbert in rather a cavalier fashion. The question of motion as a geometric process will bear further analysis from the point of view of elementary instruction. The National Committee includes a "permit to move" in its list of assumptions, and recommends its use. It is not clear to the writer, however, that such an assumption is either independent, from a strictly logical standpoint, or deducible from any set of fundamental assumptions of projective geometry which have thus far been formulated, without the use of theorems which are now generally treated as dependent upon it. The point to be made is merely this: that no mode of procedure should be tolerated in high-school geometry which is unacceptable fundamentally, but rather that any theorems seeming to demand unacceptable procedures for elementary demonstrations should be included in the list of fundamental principles accepted without proof.

Proceeding, then, from purely aesthetic considerations, to the question of what is fundamental for elementary purposes, one may tentatively propose a list of principles covering the following facts:

1\. a. The existence of a line in a plane through any two given points in the plane.

b. The uniqueness of such a line.

2\. a. The existence of a point of intersection of any two given non-parallel lines in a plane.

b. The uniqueness of such a point of intersection.

3\. a. The existence of a plane through two given intersecting lines.

b. The uniqueness of such a plane.

4. a. The existence of a line common to two given intersecting planes (i.e., two planes having a common point).

b. The uniqueness of such a line.

5. a. The existence of a perpendicular through a given point to a given line in the plane of that line and point.

b. The uniqueness of such a perpendicular.

6. a. The existence of a line parallel[1] to a given line through a given point not on that line, in the plane of that line and point.

b. The uniqueness of such a parallel.

7. a. The existence of a circle in a given plane having a given center and a given radius.

b. The uniqueness of such a circle.

8. a. The existence of a circle through three given points not on a line.

b. The uniqueness of such a circle.

9. a. The existence of a midpoint of a given segment.

b. The uniqueness of such a point.

10. a. The existence of a midray of a given angle.

b. The uniqueness of such a ray.

11. a. The existence of a segment laid off on a given line from a given point on that line, equal to a given segment.

b. The duality of such a segment (i.e., the fact that two and only two such segments exist).

12. a. The existence of an angle with a given ray equal to a given ray angle to a given angle.

b. The duality of such an angle.

13. a. The existence of a point of intersection of a circle and a line which contains a point of the interior region of the circle.

b. The duality of such a point.

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[1] Explicit statement of 2a and 6a are involved, in the familiar treatment of the subject, in the definition of parallel lines as "two lines lying in the same plane which do not meet *no matter how far they are produced.*" This modifying clause has been the source of widespread mental confusion regarding the concept of infinity as applied to geometry. As a matter of fact, parallelism, to the lay mind, is not concerned with points at a very great distance, but with identity of a quality of lines called "direction". The writer has refrained from considering the question of undefined elements, which are an obvious necessity in any logical system of definitions and theorems. He feels, however, that for the purposes of a beginner many definable terms may be added arbitrarily to the necessary list of undefined terms in order to bring them into the category of "common notions". For example, parallel lines may be *described* as lines having the same direction, in the same manner as the straight line (segment) is described by Euclid as "lying evenly throughout its whole extent".

14. a. The existence of a point of intersection of two circles whose center segment is less than the sum but greater than the difference of their radii (i.e., two circles such that the intersection point of each with the line of centers separates the center from the intersection point of the other with the line of centers, or passes through the center of the other.)

b. The duality of such a point.

15. a. The existence of a line tangent to a given circle at a given point.

b. The uniqueness of such a line.

16. a. The existence of a point common to two circles whose center-segment is the sum or the difference of the radii.

b. The uniqueness of such a point.

17. a. The existence of a triangle having two sides and the included angle equal respectively to two given segments and a given angle which is less than a straight angle.

b. The fact that two triangles having two sides and the included angle of one equal respectively to two sides and the included angle of the other, have their other sides and angles respectively equal.

18. The existence of a property of line-segments called *length,* by virtue of which any two segments (1) may be taken together additively, or subtractively, the less from the greater, to form a third, called the sum or the difference; or (2) may have a numerical ratio.

19. The existence of a similar property of angles called *angular magnitude.*

20. The existence of a similar property of closed plane figures called *area.*

21. The existence of a similar property of closed solid figures called *volume.*

22. The existence of a similar property of arcs of a circle.

23. The existence of a similar property of dihedral angles.

24. The fact that the numerical measure of the area of a rectangle is equal to the product of the corresponding numerical measures of its two dimensions.

25. The fact that the numerical measure of the volume of a rectangular parallelepiped is equal to the product of the corresponding numerical measures of its three dimensions.

26. The fact that in the same circle or in circles of equal radii two central angles have the same ratio as their intercepted arcs.

27. The fact that two dihedral angles have the same ratio as their plane angles.

If now to these facts are added the familiar "general axioms of equality" and a few "axioms of inequality" which need not be repeated here, with a statement of their application to lines, angles, areas, etc., we have a tentative basis for an elementary science which seems to be within the comprehension of the high-school student, and which may be carried to any degree of elaboration without the sacrifice of logic, but with the distinct understanding that many of these facts are themselves logically interrelated.

The first question which arises is as to the adequacy of the foregoing list (or any other) for the purposes in question. This can be answered by examining any standard list of theorems.

The second question is concerned with the ease with which these notions are grasped by beginners. This can be answered my experience only.

The third question has to do with logical dependence of certain of the items in the list upon others. The fact that there is a possible logical dependence of one item upon another which ought not be overlooked entirely should be weighed against the relative difficulty of the logical process of dependency for purposes of elementary instruction. If on consideration a decision is reached in favor of any item as it stands, as a fundamental principle, the question of its logical relationships is thereupon classified as belonging to the domain of advanced study.

The fourth question turns upon the desirability of reducing the number of items by replacing several by one more fundamental and yet equally comprehensible. This question, which is essentially not unlike the preceding, should not in any case be answered in the affirmative without a careful analysis of the deductive processes which are entailed. Efforts to reach absolute independence in laying down a set of assumptions have in the past led to acrimonious dispute and wide-spread error. Certain of the foregoing items have come to be assumed tacitly, though it is difficult to see how they can be logically dispensed with; and what is infinitely worse, and inexcusable, some have been

tacitly assumed in the proofs of the theorems, and then later proved by reference to the same theorems, a vicious circle of the worst sort.

The fifth question concerns more general concepts. It will be noted that the relationships involved in the foregoing list are restricted to equality, inequality, combination by addition or subtraction, and ratio, including numerical measure. Similarity can be defined in terms of these relations and treated logically in a manner which high-school students can appreciate, though the usual sequence in American tests involves one or more tacit assumptions. The writer has experimented with the introduction of line-symmetry (orthogonal line-reflexion) as a fundamental relation, and found apparently that many of the items given above can be demonstrated much more simply than in the usual sequence, by making assumptions to the effect that symmetric segments, angles, dihedral angles, and arcs of a circle are equal, as well as the areas of symmetric polygons and the volumes of symmetric polyhedra. A very interesting geometry can be built up in this way, which appears to be much less involved than in the familiar treatment. Congruence, for example, merely expresses the idea that two figures belong to the same symmetric sequence. Some difficulty arises in considering uniqueness from this standpoint, however, in class-room practice. There is a very serious question as to the desirability, aesthetically, of demonstrating the existence of a configuration when its uniqueness calls for an independent assumption. If this procedure is in bad taste, it may account for the awful waste of time and effort expended through the centuries upon attempted demonstrations of the axiom of parallels.

Whether or not elementary geometry should include a treatment of symmetry, there is no doubt that some terminology should be established which distinguishes carefully between point-reflexion, orthogonal line-reflexion, and orthogonal plane-reflexion, all of which are now grouped under symmetry, and lead to some very long-winded discussions in solid geometry.

Veblen has shown in a most interesting way the results of restricting equality of segments to opposite sides of a parallelogram and formulating on the basis of this relation an elementary affine geometry. It is doubtful if this procedure is practical as

a first approach to the subject, though the writer has undertaken some class-room experiments with it which have been very interesting. Its importance, of course, lies in its later application to vector analysis. It may be noted, moreover, as many have overlooked, that the general principle of "one concept at a time" seems to have been in the mind of Euclid in ordering his sequence. Legendre sought to improve on Euclid in this particular but seems to have failed to state explicitly the necessary assumptions. His effort, however, served to fix in the mind of the public the idea of economy of assumptions, which is the root of the difficulty. The writer is not quite clear in his mind as to whether redundancy of assumptions in the proof of a theorem constitutes an essential logical error. If it does, an up-to-date Euclid would have its assumptions scattered through the text as they are needed.

Euclid himself, as it is well known, tacitly assumes 14a in the proof of his Book I, proposition 1. He proves 11a with great care, but fails to mention 11b. In American texts these are both tacit assumptions, as a rule. Most American texts and many foreign texts assume 10a tacitly in proving the equality of the base angles of an isosceles triangle, and then later, as a construction, prove the same by indirect reference to the isosceles triangle. One notes with pleasure that 7b is included in the National Committee's list. It is usually not mentioned at all in modern texts.

It is strange that while 11a and 11b are by custom tacitly assumed, 12a and 12b are usually established with meticulous nicety under the guise of a "construction," in spite of the fact that 12a is prerequisite for laying down the conditions of 17b, which is usually "proved," after a fashion, and then used to prove 12a and 12b.

Statements 9b and 10b are usually regarded as covered by the "axiom," *the whole of any quantity is greater than any part of it*. This is axiomatic enough, but as a rule no proof is afforded that the "part" is a part of the whole.

It has been the custom to separate 5a and 5b each into two cases, one when the given point is on the line, and the other when it is not. It is not easy to see the reason for this, beyond the fact that Euclid makes a careful distinction between a line at right

angles *with* a given line and one which is perpendicular *to* a given line, in his constructions. One may speculate as to his motives in doing this. Is it possible that he may have considered the possibilities of an elliptic geometry of two dimensions, in which the first case of 5b would hold, while the second case of 5b would not? It is worthy of note that the National Committee has not separated these cases, while the College Entrance Examination Board enforces a separation, as has been said, by including in its syllabus the second case of 5b, which is elevated to the rank of a starred theorem. Again speculation is permissible.

It may be observed that what is usually called the "axiom of superposition" is noticeable for its absence from the list given above. Euclid omits this "axiom," but assumes it tacitly in Book I, Proposition 4. The axiom is employed solely to prove this proposition, as is well known, and it raises countless difficult questions, as to how an ideal existence can engage in a physical performance, or by what road a triangle is to travel from one place to the other, to say nothing of the necessity, on occasion, of turning a geometric idea bottom-side-up. It would seem a better procedure to include the proposition in the list of fundamentals (17b) after the manner of Hilbert, and to leave these questions to the metaphysicians. In any event it is hardly profitable to concoct an "axiom" for the sake of proving a single theorem, particularly when the proof involves another "axiom" (1b) of which the theorem is really independent.

A somewhat more serious situation, historically, arises with reference to 6a. Hilbert includes it in his list of assumptions, while Veblen omits it from his list for the excellent reason that he proves it as a theorem. From the point of view of general geometry this statement is of course highly important in contrasting Euclidean and elliptic geometry. This, however, is a consideration beyond the abilities of the beginner. The question, nevertheless, arises very early in the high-school course. Proof by reference to 1b or to 5b by way of 17b, which is the usual American custom, is sound logically, but is generally regarded by students as difficult.

In 6b we recognize our old friend the "parallel axiom," which distinguishes Euclidean from hyperbolic geometry. We are all well aware of the futility of engaging in attempts to prove this

"axiom" by familiar processes. Some of the efforts to prove 8b are not much better, though it is probably going too far to say that they are all false.

Statements 18-27 are intended to relegate the whole discussion of measurement to to the domain of "Foundations." It is difficult to believe that any teacher of high-school experience would object to this procedure.

It may be noted that 24 and 25 are usually printed nowadays as "assumptions," displacing the somewhat ridiculous "theorem of limits" of an earlier era.

If the statements usually listed as "Problems of Construction" in text-books and syllabi are intended merely as applications of geometry to mechanical drawing, it is not our purpose to consider them here, beyond noting that draftsmen usually employ methods other than those printed in geometries. If on the other hand the constructions are regarded as existence theorems, and therefore as essential steps in the sequence, they amount to an attempt to prove that all of the other principles of plane geometry mentioned above are deducible from 1a, 1b, 6b, 7a, and 7b, with the "general axioms." We now know that Euclid, in spite of the amazing superiority of his logic, failed to do this. His successors have done worse.

One principle might well be added to the foregoing list, viz., Cavalieri's theorem for volumes of solid figures. It is well-known that some further assumption independent of the foregoing list is necessary to prove the formula for the volume of a pyramid. This takes various forms in elementary texts. But it is doubtful whether the question of the relative merit of this or that assumption is of value to the beginner, either in school or college. Cavalieri's principle shortens the whole treatment of volumes of oblique prisms and of pyramids. The general proof of it, by elementary methods, is rather too difficult for the student who is not specializing in mathematics. It comes near to being "self-evident."

It will be noted also that there is no reference in the foregoing list to the rectification or determination of the area of a circle, or to the areas of curved surfaces or the volumes of "round bodies." The usual "proofs" of theorems concerning these topics are, as is well known, erroneous. It would be well

if professional opinion could be brought to decide definitely as to whether these questions should be left to the calculus for consideration, or, on the other hand, the theory of limits should be introduced into the subject late in the course, employing a logically correct treatment. The present state of the case is unsatisfactory. Particularly so is the usual statement, "A line-segment is the shortest distance between two points," in advance of any theory of rectification, when distance can be defined only as the numerical measure of the line-segment itself. The whole treatment of the so-called "incommensurable case" is in most texts perforated with error. High school students of geometry are now-a-days familiar with irrational numbers, though the treatment in elementary algebra texts is often obscure. But the "incommensurable case" does not help the situation any.

Euclid's treatment of ratio and proportion is, as Dean Fine has pointed out, his most signal achievement, illustrating in singular fashion the remarkable analytic faculty of the ancient Greeks. But in the light of modern algebra this treatment is out of place in an elementary course, and there is no excuse for dabbling with it. The fundamental error in proofs of rectification, etc., as given in elementary texts lies in the tacit assumption that as one configuration appears to approach another in space, their numerical measures approach equality. The trouble with this tacit assumption is that it is false.

The whole problem of elementary geometry would gain in clarity from a careful study of the relation of the subject to physics. One may stand solidly on the ground that geometry is a branch of physics, and that research into its logical foundations is a waste of time. Possibly this view is best for elementary purposes, but it leaves in the air many questions of criteria of logical validity. Until such time as physicists and astronomers are able to decide as to the character of universal space, it is likely that mathematicians will go on trying out the consequences of this or that set of assumptions. If agreement could be reached by common consent as to a *workable* set of assumptions and a *useful* list of undefined elements, all that lies back of that logical boundary could be treated experimentally or logically, as one might choose. Fundamental questions as to a universal geometry would not then appear to conflict with the truth, as

they often do at present. One is sometimes tempted to throw overboard the bulk of our traditional procedure and to draw fundamental geometrical distinctions axiomatically from the right triangle, thus changing elementary geometry into the analytic form, as Veblen has somewhere suggested. This would afford a wonderful economy of time, but popular approval is doubtful, to say the least.

The content of the foregoing list is suggested as suitable material for work in Informal Geometry mentioned in the report of the National Committee. It is recommended that the title "Intuitive Geometry" be dropped entirely, as it conveys the erroneous impression that intuition can be dispensed with in connection with logical deduction.

MATHEMATICAL RECREATIONS

By MARTHA PIERCE

There are certain instincts inate in every one of us, the instinct of curiosity and the instinct of the self or the ego-instinct, which give us that increasing desire to solve puzzles. We love to be confronted by a mystery and we are not entirely happy until we have solved it, even though the only reward of our work may be the pleasure derived from the knowledge that we ourselves have reached the solution. The spirit of rivalry stimulates every person to solve the puzzles which come to him and keep on a level with his companions.

The solution of puzzles brings other rewards as well. It[1] stimulates the imagination and develops the reasoning power. We often learn little tricks that are great time-savers in our later work. In fact Fitzosborne considered the art of making and solving puzzles to be a necessary acquirement of all, an art which might well be used in universities to convey some of the most useful principles of logic. One of his maxims was "He who knows not how to riddle knows not how to live."

It is interesting to consider how puzzles containing an original idea are invented. We are told that one cannot merely sit down and invent a good puzzle to order. Notions for puzzles are suggested by something we see or hear, and are led up to by other puzzles that have come to our notice. Often times a new idea comes from the blunder of somebody over another puzzle. Puzzles can be made out of almost anything, once the person has the idea. Coins, matches, cards, counters, bits of wire or string, any of these can be used.

Two very general classes of puzzles may be called (1) those built up on some interesting or informing principle and (2) those which conceal no principle whatever such as a picture cut at random into bits to be put together again.

The ancient riddle that arouses the imagination is a type of[2] puzzle coming under the first class. A good example of this is

[1] Dudeny, "The Canterbury Puzzle"; pp. 12-22.
[2] Dudeny, "The Canterbury Puzzles"; p. 16.

the riddle of the Sphinx of Boeotia. "What animal walks on four legs in the morning, two at noon, and three in the evening?" This riddle the Sphinx asked of all the inhabitants of her kingdom and of all travelers who ventured to that place. When they failed to give the right answer she devoured them, one after the other. But one day when a certain traveller gave the answer "Man" to her riddle, the sphinx dashed herself to pieces against the cliffs, for she knew that she was conquered. Today this type of riddle exists as the conundrum based on a play on words as "When is a door not a door?"

Next we have the letter puzzles based on peculiarities of[1] language such as anagrams, word-squares, and charades, puzzles very similar to the modern cross word puzzle. These are very ancient forms of puzzles, especially those involving words and sentences that read backwards and forwards alike. We even have the story that Adam introduced himself to Eve saying, "Madam, I'm Adam," to which his companion replied "Eve."

The arithmetical puzzles[2] range from simple equations to complicated problems in the theory of numbers. From the earliest book on arithmetic to be printed in England "The Ground of Arts" by Robert Recorde we find that the multiplication table was learned only as far as five times five. For multiplying together any numbers greater than these they used a very ingenious scheme. Place the numbers, such as 6 and 8, on the left

$$\begin{matrix}8\\ 6X\end{matrix}$$

hand side of a large letter X[1] as 6*X*. Subtract each from ten and write the results directly opposite on the right hand side. Then

$$\begin{matrix}8\ \ 2\\ 6\ X4\end{matrix}$$

draw a line under the whole, as 6 *X*4. The units figure of the product is obtained by multiplying together the two remainders, and the other figure by subtracting either remainder from the number crosswise from it.

In the seventeenth century in England multiplication was carried out by use of Napier's bones—nine sticks numbered at the top 1 to 9 inclusive, each stick having on its side the first nine multiples of the number at the top. When it was desired to multiply a number by 57, such as 89, the sticks headed 5 and 7

[1] Dudeny, "The Canterbury Puzzles"; p. 17.
[2] Licks, "Recreations in Mathematics"; p. 9.

were taken and their multiples used. First the multiples of 5 by 8 and 9 were set down as $\begin{matrix}40\\45\end{matrix}$. Then the multiples of 7 by 8 and 9 were set down in a similar fashion, and the addition gives the product.

$$\begin{array}{r}
45\\
56\\
63\\
\hline
5073\\
89\\
57\\
\hline
40
\end{array}$$

About 1750 when Ozanam's "Recreations Mathematiques" was published at Paris, discussion was revived concerning perfect numbers, numbers which are equal to the sum of their divisors. Such numbers are $6 = 1 + 2 + 3$, $28 = 1 + 2 + 4 + 7 + 14$, etc. As far as is known all perfect numbers end in 6 or 28 and an odd number cannot be perfect.

To find a number selected by someone is a common arithmetical[1] pastime which has numerous methods of solution derived from simple algebra, provided that the result of certain operations on the number is known. Other recreations of the same type are to find the result of a series of operations performed on any number (unknown to the operator) without asking any questions, and to find which of two numbers, one even and the other odd, have been selected by each of two persons. Each of these problems is solved by performing the given operations on an unknown and then when certain results are given, solving the simple equation and obtaining the value of the unknown. Similar methods are used by entertainers to determine the dice thrown at random by a boy in the audience or to find the age of a person in the audience.

Various kinds of arithmetical problems have appeared for centuries in nearly every collection of mathematical recreations. We have the type illustrated by[2] "A man goes to a tub of water with 2 jars, of which one holds exactly 3 pints and the other 5

[1] Ball, "Math, Recreation and Essays"; pp. 4-11.
[2] Ball, *Mathematical Recreations and Essays;* ch. I.; pp. 18-27.

pints. How can he bring back exactly 4 pints of water?" Exploration problems have always been popular, problems concerned with the maximum distance into a desert which could be reached from a frontier settlement by the aid of a party of explorers, each capable of carrying provisions that would last one man for a day. The decimation problem most often takes the form "A ship carrying[1] as passengers 15 Turks and 15 Christians encountered a terrific storm and the pilot declared that in order to save the ship and the crew one half of the passengers must be thrown into the sea. To choose the victims the passengers were placed round in a circle, and it was agreed that every ninth man should be cast overboard, reckoning from a certain point. By simple counting an arrangement can be obtained by which all the Christians would be saved."

Numerous arithmetical fallacies have stimulated the minds of[1] many persons for centuries, such as

$$\begin{aligned} (-1)^2 &= 1 \\ 2\log(-1) &= \log 1 = 0 \\ \log -1 &= 0 \\ -1 &= e^{\circ} \\ -1 &= 1 \end{aligned}$$

Prominent among the fallacies of algebra is the fallacy of employing a process that is not uniquely reversible. The "fallacy of accident" by which one argues from a general rule to a special case where some circumstance renders the rule inapplicable, and its converse fallacy, and the fallacy from one special case to another, all find exemplification in pseudo-algebra. As a general rule if equals be divided by equals, the quotients are equal, but not if the equal divisors are any form of zero. For example, suppose $a = b$. Then

$$\begin{aligned} ab &= a^2 \\ ab - b^2 &= a^2 - b^2 \\ b(a-b) &= (a+b)(a-b) \\ \therefore\ b &= a + b \\ b &= 2b \\ 1 &= 2 \end{aligned}$$

Other algebraic fallacies are based on the improper use of the quantity o/o, or neglect of the fact that a quantity has two square roots, one positive and one negative.

[1] White, "Scrapbook of Elementary Math."; pp. 86-87.

Cajori in his "History of Elementary Mathematics" gives us the[1] problem of determining the age of Diophantus from his epitaph. "Diophantus was a child for $1/6$ of his life, a youth for $1/12$, and a bachelor for $1/7$; five years after his marriage a son was born who lived $1/2$ as long as his father and who died four years before his father."

We also have the problem which is said to have caused the death of Homer through his vexation at not being able to reach its solution. One day as he was walking along the shore he asked of certain fishermen how many fish they had caught. It was this reply, "As many as we caught, we left, as many as we did not catch, we carry" which caused him so much worry.

Geometric amusements are countless in number, such as the proof that a right angle equals an angle which is greater than a right angle and the proof that every triangle is isosceles. One of those handed down from the Middle Ages was expressed in poetry,

"A castle wall there was, whose height was found
To be just 50 feet from top to ground.
Against the wall a ladder stood upright,
Of the same length the castle was in height.
A waggish fellow did the ladder slide,
The bottom of it 5 feet from the side.
Now I would know how far the top did fall
By pulling out the ladder from the wall?"

A familiar problem is the ferry boat problem of which a common statement is "A showman was travelling with a wolf, a goat, and a basket of cabbages. For[2] obvious reasons he was unable to leave the wolf alone with the goat or the goat alone with the cabbages. During their journey the party arrived at the bank of a river across which the only means of transportation was a boat so small that the showman could take with him but one thing at a time. How was the passage effected?"

A proof has been devised to show that every ellipse is a circle. The focal distance of a point on an ellipse is given in the usual notation in terms of the abscissa by the formula $r = a + ex$. Hence $\dfrac{dr}{dx} = e$. From this it follows that we cannot have a maximum or a minimum value. But the only closed curve of

[1] Licks, "Recreations in Mathematics"; pp. 32-33, 48.
[2] *Mathematical Recreations and Essays;* ch. IV.

which the radius vector has not a maximum or minimum value is a circle. Hence every ellipse is a circle.

Paradoxes and fallacies decrease as we ascend the mathematical ladder. Under Arithmetic there are many and under Algebra about the same number. Under Geometry we have a few, but under Trigonometry only one can be found, and under Analytic Geometry none at all.

About 1880 everyone in Europe and America was engaged in[1] the solution of the Fifteen Puzzle, invented by a deaf and dumb man as a solitaire game. A square shallow box contained fifteen blocks numbered 1 to 15 inclusive and these could be moved about one block at a time on account of one empty space. The blocks being placed in the box at random the problem was to arrange them in regular order. Sometimes it happened that when the last row was reached the block marked 15 preceded that marked 14, and then the player would work for days at a time to get the blocks in right order. This puzzle was the cross word puzzle of that day. Everybody worked at it in all sorts of places. People in public conveyances could be seen working at their puzzles. Physicians were arguing as to whether it was beneficial mental exercise or a cause of nervous disorders. However when mathematical analysis proved that when block 15 preceded block 14 in the last row, it was impossible to reach the correct solution and that one-half of the random arrangements of the blocks would result in that state of affairs, the craze gradually abated and the Fifteen Puzzle was abandoned.

Perhaps one of the most popular mathematical recreations is[2] the well-known magic square which consists of a number of integers[3] arranged in the form of a square, so that the sum of the numbers in every row, column and diagonal is the same. The simplest square contains nine digits which may be arranged in eight different ways such that the sum of each row, column, and diagonal is 15. A true magic square should have[1] for its smallest number and contain all the natural numbers up to n^2 the sum of which is $1/2n^2\ (n^2 + 1)$. Following these rules and others worked out in relation to them we see that magic squares of any desired order may be constructed.

[1] Licks pp. 20 and 21.
[2] Ball, pp. 137-163.
[3] Licks, pp. 39-43.

The formation of these squares is an old amusement, and in times when mystical philosophical ideas were associated with particular numbers, it was natural that such arrangements should be considered to possess magical properties. Magic squares of an odd order were constructed in India before the Christian era according to a known law of formation. They were introduced into Europe by Moschopulus of Constantinople in the first part of the fifteenth century. The majority of the medieval astrologers and physicians were greatly impressed by such arrangements. Cornelius Agrippa (1486-1535) constructed magic squares of orders 3, 4, 5, 6, 7, 8, 9 which were associated respectively with the given astrological bodies then called "planets," Saturn, Jupiter, Mars, the Sun, Venus, Mercury, and the Moon. He taught that a square of one cell, in which unity was inserted, represented the unity and eternity of God; while the fact that a square of the second order could not be constructed illustrated the imperfection of the four elements, air, earth, fire, and water. Later this fact was taken to be symbolic of original sin. A magic square engraved on a silver plate was sometimes prescribed as a charm against the plague. Such charms are still worn in the East.

We have two forms of what are known as hyper-magic squares, (1) pandiagonal squares which are magic along the broken diagonals as well as along the two ordinary diagonals, and (2) symmetrical squares which are of order n and are so constructed that the sum of any two numbers in cells geometrically symmetrical to the center is constant and equals $n^2 + 1$. Doubly-magic squares are those of order n so made up that if the number in each cell is replaced by its square the resulting square will also be magic. Trebly-magic squares have been constructed, that is, squares which are magic for the original numbers, their squares and their cubes, but there are none of an order lower than 128. Benjamin Franklin, the famous philosopher and diplomat amused himself with magic squares. He devised a square of 2,056 cells which is called "the magic square of squares."

The solution of magic stars is a problem that grew directly from that of magic squares. This deals with the situation where a re-entrant octagon is constructed by the intersecting sides of two equal concentric squares. It is required to place the first sixteen natural numbers on the corners and points of intersection

of the sides so that the sum of the numbers on the corner of each square and the sum of the numbers on every side of each square equals 34. There are also magic circles, rectangles, crosses, diamonds, cubes, cylinders, and spheres.

An interesting mathematical discovery of recent date is that[1] only four different colors are necessary in order to color the most complicated map of a country so that contiguous sides of districts shall not have the same color. The reason for this is that it is impossible to draw five areas so that a boundary of each shall be contiguous to the other four, taking the word contiguous to mean that the areas border along a line, not at a point. Thus we get the solution of the four color's problem.

Several of the unicursal problems are known the world over,[2] and are constantly being worked upon and extended. Euler's Problem brings in the discussion as to whether it was possible to take a walk in the town of Königsberg in such a way as to cross every bridge in the town once and only once. The town at the time of Euler had seven bridges and included the island of Kneiphof. Euler not only solved that problem, giving the solution that such a journey is impossible at one time, but can be completed in two separate routes, but he discussed the general problem of any number of islands connected in any way by bridges. The problem in its solution was reduced to finding whether a given geometrical figure can be described by a point moving so as to traverse every line in it once and only once.

Another type of the unicursal problem is the maze, of which only a few exist at present, one of the most notable being at Hampton Court. The problem of the maze was also solved by Euler, following directly from his former work. One class of ancient mazes consisted of any complicated building with numerous[3] vaults and passages, really a labyrinth. Another class was a winding path confined to a small area and leading to a shrine in the center. It was this latter form which was built for the Minotaur and which was traced out on the backs of the coins of Cnossus. Copies of the maze of Cnossus were frequently engraved on Greek and Roman gems; similar but more elaborate designs are found in numerous Roman mosaic pavements. A copy of the Cretan labyrinth was embroidered on many of the

[1] White, "Scrapbooks of Elementary Mathematics"; pp. 140-141.
[2] Ball, "Mathematical Recreations and Essays"; pp. 170-181.
[3] Ball, "Mathematical Recreations and Essays"; pp. 82-189.

state robes of the later Emperors, and was thence copied on the walls and floors of various churches. At a later date in Italy and France these mural and pavement decorations developed into scrolls of great complexity consisting always of a single line. When pictured on the floors of churches it is thought they were used to represent the journey through life as a kind of pilgrim's progress.

In England the mazes were usually out in the turf adjacent to some religious house; and it is said that the monks traversed this maze as a religious exercise, repeating a prayer at each turning. The modern maze was introduced from Italy during the Renaissance, and many of the palaces and mansions built in England during the Tudor and Stuart periods had labyrinths attached to them, as the royal palaces of Southwark, Greenwich, and Hampton Court.

The Hamilton problem, so-called because it was solved by[1] Sir William Hamilton, is just the reverse of the Euler problem. Here the problem is to go "all around" the world, that is, starting from any town to go to every other town once and only once and to return to the initial town, the order of the n towns to be first visited being assigned, where n is not greater than five.

The Kirkman's School-Girls problem is one, the general theory[2] of which has interested mathematicians for years. This problem states that a school-mistress was in the habit of taking her girls for a daily walk. The girls were 15 in number, and were arranged in five rows of three each, so that each girl might have two companions. The problem is to dispose of them so that for seven consecutive days no girl will walk with any of her school-fellows in any triplet more than once. The general statement of the problem requires the arrangement of n girls, where n is an odd multiple of 3, in triplets to walk out for y days, where $y = \frac{(n-1)}{2}$, so that no girl will walk with any of her school-mates in any triplet more than once. This problem has had countless solutions and extensions as n has been given different values.

We also have handed down to us the story that the temple[3] of Benares, the dome of which marks the centre of the world,

[1] Ball, "Mathematical Recreations and Essays'; pp. 189-192.
[2] Ball, "Mathematical Recreations and Essays ; pp. 193-223.
[3] Ball, "Mathematical Recreations and Essays"; p. 229.

rests a brass plate in which are fixed three diamond needles, each a cubit high and as thick as the body of a bee. On one of these needles there was placed at the creation sixty-four discs of pure gold, the largest disc resting on the brass plate, and the others getting smaller and smaller up to the top one. This is the tower of Brahma. Day and night without pause the priests transfer the discs from one diamond needle to another according to the fixed laws of Brahma which require that the priest must not move more than one disc at a time and that he must place this disc on a needle so that there is no smaller disc below it. When the sixty-four discs shall have been thus transferred from the needle on which they were originally placed, the tower and the temple will crumble into dust, and with a thunder clap the world will vanish. The question is how do the priests thus move the discs and how long will it take them to complete their task. Mathematicians have figured that it would take thousands of millions of years.

The puzzle of the camels is also interesting and entertaining.[1] There was once an Arab who had three sons. In his will he bequeathed his property, consisting of camels, to his sons, the eldest son to have $\frac{1}{2}$ of them, the second $\frac{1}{3}$, and the youngest $\frac{1}{9}$. The Arab died leaving 17 camels, a number not divisible by 2, 3, or 9. As the camels could not be divided, a neighboring sheik was called in consultation. He loaned them a camel, so that they had 18 to divide. The first son took $\frac{1}{2}$ or 9, the second $\frac{1}{3}$ or 6, and the third $\frac{1}{9}$ or 2, making 17 camels in all. They had divided according to the will, and were able to return the camel that had been loaned to them. It should be noted, however, that $\frac{1}{2} + \frac{1}{3} + \frac{1}{9} = \frac{17}{18}$, and not unity.

All of these problems and puzzles mentioned above may be put in one definite class, based on the fact that all can be solved and have been solved. Many have been extended, and have numerous solutions for their various extensions. Nevertheless there are problems in existence today, which have never been solved, and which have now been proved insoluble; the three famous problems of antiquity, the tri-section of the angle, the duplication of the cube, and the squaring of the circle. The tri-

[1] White, "Scrapbook of Mathematics"; p. 193.

section of the angle is a very ancient problem, but as Ball expresses it "tradition has not enshrined its origin in romance."

The problem of the duplication of the cube was known in[1] ancient times as the Delian problem. The story is told that the Athenians once suffered from an extensive plague, and so they finally consulted the oracle at Delos to find out how to stop it. The god of the oracle, Apollo, replied that they must double the size of his altar which was then in the form of a cube. The Athenians set to work and soon had completed the construction of an altar each of the edges of which was double that of the old one. When they came to the god and told him that his commands had been obeyed, he was very furious and made the pestilence worse than before. He demanded for his altar a cube, the volume of which was twice that of his old one, and not eight times as large, as was the altar they had built. The Athenians then applied to Plato who referred them to the geometricians of the time. To their great sorrow these people were then told that the problem had never been solved by use of the ruler and compass, the instruments with which they had to work.

Modern mathematics has proved these three problems impossible of solution with ruler and compass alone, and has shown new ways of solving them, if that limitation be abandoned. However, it was not until 1882 that the transcendental nature of π was established by Lindemann. Many methods of approximation to the value of π have been devised and applied. One man has even computed its value accurately to 707 decimal places. The following rhyme enables one to easily remember its value to twelve decimals, the number of letters in each word corresponding to the integers in the value of π:

"See I have a rhyme assisting[2]
My feeble brain its tasks
Sometimes resisting."

The quadrature of the circle has been the most fascinating of the three problems. In each generation all down through the ages hundreds of persons have attempted to give solutions for it, people of all sorts and descriptions. We have the story that an agricultural laborer squared the circle and brought the problem

[1] White, "Scrapbook of Mathematics"; p. 122.
[2] Lick, *Recreations in Mathematics;* ch. III.; p. 50.

to London. He left his papers with De Morgan, the famous mathematician of the day, together with a copy of a letter to the Lord Chancellor desiring 100,000 pounds the amount of the alleged offer of reward for the solution of the problem. When[1] told by De Morgan that he did not have the knowledge requisite to see in what the problem consisted, the quadrator wrote to De Morgan telling him he should "change his business and appropriate his time and attention to a Sunday School to learn what he could, and to keep the little children from dirtying their clothes." In 1755 the French Academy of Sciences came to the conclusion not to examine any more quadratures or kindred problems, and a few years afterward the Royal Society followed the same course. But even today enthusiasts are at work on these problems and their solutions. In 1905 a book called "The Secret of the Circle and the Square" was published in Los Angeles, in which the tri-section of the angle was also considered and new methods of approximation to the solutions of the problems were given. We frequently hear of high school pupils who firmly believe they have solved one of the problems and publish their results abroad.

Even though these problems have been proved insoluble the work on them by so many mathematicians has not been in vain, for the by-products of this work have been many. It has led to all sorts of important discoveries, and has also greatly quickened interest in mathematical questions.

And so one might go on considering problem after problem, each interesting, informing, and worthy of solution and extension. Throughout the ages people have found mathematical recreations a great source of amusement, and today a countless number of persons are engaged in making and solving puzzles, problems, and their extensions with just as much interest and enthusiasm as in the preceding centuries. In fact, so broad and extensive is this field that one might almost choose as his profession the making and solving of mathematical puzzles and problems.

[1] De Morgan, "A Budget of Paradoxes"; vol. I; p. 163.

RULE AND REASON IN ALGEBRA

By PROFESSOR RALPH BEATLEY
Harvard University

Miss Young: "Miss Elder, how do you teach — 5 times — 3 equals + 15? I have taught algebra for three years now and a lot of my pupils have great trouble in seeing just why — 5 times — 3 gives + 15. They see perfectly well that the rule 'minus times minus gives plus' seems to work out all right and they are able to follow the rule in doing their exercises. But I have a feeling that some of the more conscientious among them who are really trying to understand what it is all about are not convinced by my explanations, and that they are beginning to follow rules just so as to get the right answer and to please me. It seems to me that unless I can replace this attitude by a more comprehending one on their part, they will fail to get much of the good which the study of algebra should give them, and will build up a wrong attitude toward mathematics in general.

"I have explained very carefully that negative numbers can represent losses and debts and that in character they may be regarded as opposite in sense to positive numbers. In fact, in my efforts to have them see just why minus times minus should give plus, I have made so much talk about successive subtractions of negative lengths and the successive cancellations of debts to improve my financial standing, that I really believe my pupils have the idea by now that this particular operation *is* hard to justify and that it is probably beyond their powers of comprehension. I shouldn't care so much about this particular topic did I not feel that some of my pupils who are trying hardest to understand the subject will soon give up in despair and go to swell the army of those who simply follow the rules and 'see what case it comes under.' I certainly don't want that to happen. I don't want them to lose confidence in their ability to reason. I want them to have the experience of reasoning successfully so often that they will grow to expect to understand a new topic or a new process and will become confident in their own powers.

"I try to get them to reason and to understand *at every step*, and I think they are doing better than they would if I were teach-

ing in the old formal, cut-and-dried, way with its great emphasis on technique. But, as I say, I am not satisfied, and I believe that something is wrong somewhere."

Miss Elder: "Your experience reminds me of my own troubles when I first taught algebra. I used to indulge in involved explanations to justify the operations with negative numbers; but I soon came to realize—as you have realized—that I was imparting an air of mystery to the subject that did not properly belong to it. So with a somewhat guilty feeling I decided to present this somewhat troublesome topic to my next class in a casual way, and very briefly, simply showing my pupils that these new operations seemed not to violate any of their previous notions and asking them to accept the rules and to use them whenever occasion demanded. Some of the results of these rules seemed to have plausible enough interpretations in terms of debts and so forth, but these interpretations we regarded simply as illustrations and applications which exemplified the rules.

"I must confess that I am no more satisfied than you with the results of my teaching. For I cannot see that my method of handling this particular topic helps my pupils to reason; I simply have given up trying to make it.

"I have comforted myself somewhat with the doctrine that technique has some considerable value in itself, in that through constant practice pupils come gradually to a realization of the reasoning which underlies the technique. For example, I feel very sure that when my pupils first learned long division in arithmetic a lot of them may very well have understood at the time the reasoning which justified the process. But I feel that a real comprehension of long division was gained only by those pupils who did many, many examples during the succeeding years and from time to time considered what they were doing while they were doing them. I believe that this is strictly true of long division and of certain operations with fractions. I cannot see, however, that it is necessarily true also of negative numbers.

"My present method of teaching negative numbers rests, therefore, on the following very unsatisfactory foundation: that it is psychologically sound to teach *some* topics on the 'it works, so I guess the process is correct' basis, and that for some topics

also it is correct to trust in a gradual comprehension through practice in technique. But whether it is legitimate to treat negative numbers in similar fashion, I do not know. I agree with you, too, that it does not seem right to ask them to take these rules on faith. That seems to me to be contrary to the whole spirit of mathematics.''

Here are two conscientious and thoughtful teachers trying to do the right thing for their pupils and yet dissatisfied with their results. Would not Miss Young be helped in her efforts to encourage her pupils to reason, if she were reminded at this juncture that not everything in mathematics can be proved; that reasoning in mathematics must begin somewhere—with certain undefined notions, definitions, and assumptions; and that the topic of negative number is one of these starting points? That she does well to encourage her pupils to reason wherever possible, but that this is one of the few places where it is *not possible*. And would not Miss Elder be happier in her procedure of 'this is the rule; follow it,' if she realized that the best mathematicians can do no better?

There are certain definitions and assumptions which underlie the subject of algebra. That this is so with respect to geometry is familiar to all. It is equally true of algebra. In the effort to make subtraction always possible, to give to $a - b$ a meaning even when b is greater than a, our mathematical forebears invented the concept of negative number and devised rules to govern operations therewith. Other rules could have been devised, just as in the field of geometry we can have other systems of geometry than that of Euclid. It was not necessary to have laid down the rules "— 7 times 3 equals — 21" or "— 5 times — 3 equals + 15." Why did they choose these rather than their opposites? Simply because negative numbers were invented to fill certain needs and so the rules governing operations with negative numbers were devised so as to satisfy these needs.

We can add or multiply any two (positive) numbers regardless of their size; and with the help of fractions—which were devised to that very end—we can divide any two (positive) numbers, except that we may not divide by zero. It is very awkward, therefore, to have to be mindful always of the size of the numbers when we are subtracting. We do not like excep-

tions; we like to generalize as much as possible. 5 from 7 leaves 2; 6 from 7 leaves 1; 7 from 7 leaves 0; 8 from 7 leaves "it can't be done." Isn't that annoying? Let us call the remainder -1 and say that this subtraction may be checked by addition in the same way that $5+2=7$ checks $7-5=2$: that is, that $8+-1$ shall equal 7. Just by imperial decree we have invented a new number, -1, and two new rules, $7-8=-1$ and $8+-1=7$. These can now be illustrated by the familiar interpretations, "I have 7 dollars, but I owe 8 dollars; I am 1 dollar to the bad" and "A gain of 8 and a loss of 1 amounts to a net gain of 7" respectively. That the rules can be so interpreted is not strange since they were invented to make this very thing possible.

We tell our pupils that in solving equations they may add the same number to each side of the equation, and similarly for subtraction, multiplication, and division; and we usually do this before discussing negative numbers with them. We set the example $13n+2=7n+6$ and expect them to arrive at the result $n=\frac{2}{3}$ by means of the intermediate stages $13n-7n=6-2$ and $6n=4$. Most of them do. But what shall we say to the pupil who in perfectly good faith subtracts $13n$ from each side instead of $7n$, and 6 instead of 2, and is unable to proceed beyond the next step $2-6=7n-13n$? What is there in the spirit of the rule that makes it necessary to be mindful of the size of the numbers subtracted? Here again it is awkward not to have the same freedom in subtracting that we have with the other operations. Why should the axiom about subtracting equals from equals have to be different from the others?

Why not say to this pupil, "It cannot be done this way right now; but, so far as you know, what you have done is quite correct. A little later we shall see that in order to get around just such awkward little situations as this we shall be permitted to continue your work as you have begun it and to say $-4=-6n$; $\frac{-4}{-6}=n$; $\frac{2}{3}=n$, getting the same answer as the others. If we were permitted to say $2-6=-4$ and $7n-13n=-6n$; and if we had authority to divide -4 by -6 and get $\frac{2}{3}$ for an answer, then we should not need to consider the size of numbers when subtracting and we could solve equations without being embarrassed by awkward situations such as this. Almost the next thing we shall do will be to establish rules of this sort, and

then we shall consider your method of solving this equation to be just as correct as the method the others used."

In all of the above there is no intention to decry the illustrations and interpretations which teachers commonly give to negative numbers, other than to call attention to the fact that the illustrations will not bear the strain which is frequently put upon them. For most pupils such illustrations—to the extent at least that they are understood—are all the proof desired. They are not proofs, however, despite the fact that they carry conviction to some. So long as the doctrine "it gives the right answer, so the rule must be right" satisfies the pupil, it is appropriate and helpful to depend on it. As we lead them to give reasons for every step, however, and they in turn begin to press us for more and better reasons, it behooves us to be ready with a more mature answer when they burst the chrysalis and start to look around them.

For those pupils or teachers who cannot see or thoroughly grasp the inevitableness of the relation $-5 \times -3 = +15$ it should be comforting to know that they are not expected to, and that no one can. With this in mind, our efforts to encourage pupils to reason should be the more fruitful for not expecting them to reason in situations where it is not possible to do so.

THE TEACHING OF MATHEMATICS IN AN ENGLISH SECONDARY SCHOOL[1]

By MARGARET BROWN
The Bishop Auckland Girls' County School

Whilst the work of pioneers in education is generally known in other countries and the work of exceptional schools is often watched with interest, teachers both in America and in England are singularly ignorant of what is the general practice in the schools on the other side of the Atlantic. Hence, the organization of the mathematical work in a common type of English secondary school may be of interest to American teachers of mathematics.

The English Board of Education requires that a Secondary School shall offer a progressive and complete course of study of not less than four years (from 12 to 16), including English Language and Literature, a foreign language, history and geography, mathematics, science and drawing. In addition to these subjects, it is customary to teach also scripture, physical exercises, singing and some manual or domestic work. In practice, our secondary schools offer courses of from four to ten years' duration.

The Bishop Auckland Girls' County School, in which I teach, offers a six years course (12 to 18). It is a school which corresponds as closely as any type of English school can do, with the American Public High School. The school is financed partly by the fees paid by the pupils, partly by the national exchequer, and partly by an education rate raised locally. All schools receiving state aid must have at least 25% of free scholars, and are subject to inspection of the Board of Education Inspectors. This particular school has 60% of free scholars. There are about 400 scholars and one teacher is allowed for every 20 pupils, in addition to the principal. This is considered a large school in England, the usual number of pupils being about 300.

Entrance to the school is obtained as the result of a competitive examination. The examinees are usually pupils from the

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[1] Presented at the Spring Meeting of the Association of Teachers of Mathematics in New England, May 2, 1925.

surrounding primary schools, aged 11-12½ years, but others are also eligible for entrance. The examination comprises a paper in Arithmetic, one in English, and an oral examination. About 400 or 500 candidates present themselves for about 70 vacancies. Competition is not so keen as this in all districts, but there is a great shortage of secondary school accommodation generally. Competitors on the list who are next those obtaining free places may become fee-paying pupils. Usually these enter the school, but they are sometimes unable to do so for economic reasons. Fees in schools of this type vary from about £5 to £20 per annum in various parts of the country, according to the progressive or non-progressive character of the local education authority. Parents of pupils entering this particular school are required to sign an undertaking to keep the pupil in school until at least the age of 16.

Classes number 35 pupils and are called Form I, Form II, etc., indicating how long the pupil has been in the school. For certain subjects forms are grouped together and redivided according to progress in that particular subject, notably in mathematics and in languages.

The school day is divided into seven periods of 35 or 40 minutes. It is unusual to give study periods in school, except in Form VI. No subjects are elective, but every pupil takes the full course offered by the schoool. All subjects are studied continuously for the five years' course ending with the First School Leaving Examination. Only a small percentage of the pupils stay on at school to take the Second School Leaving Examination, which is highly specialized and exempts pupils from a part of a university course. The First School Leaving Examination is not primarily intended for a College Entrance Examination, but if a pupil passes with sufficient credit in the various subjects required for college entrance, all at one time, then exemption from the Matriculation Examinations of various universities can be obtained.

The time allotted to mathematics is five periods weekly for the first five years. In the sixth form the time allotted depends on whether mathematics is being taken as a principal subject or a subsidiary one in the Second School Leaving Examination, or there may be none at all. Pupils do home work three times

weekly, fifteen minutes each time in Form I and increasing to thirty minutes each time in Form V. This means that almost all the work is done in school under the personal supervision of the teacher. Home work is regarded rather as a test on or recapitulation of the days' lesson rather than as preparation for the following one.

In Form I pupils are divided into A, B, and sometimes C divisions for mathematics according to their marks obtained in arithmetic in the entrance examination. The number in each division varies from 20 to 35. Classes for all divisions are held at the same time and after every examination pupils may be drafted into upper or lower divisions according to progress, so that each class has pupils of fairly equal ability and attainments. A teacher keeps the same division for five years if practicable, but the same teacher does not always get upper or lower divisions; teachers take upper and lower divisions in turns.

In describing the work done I usually refer to the work of a B division. These pupils will normally pass the First School Leaving Examination in Form V. Pupils in the C division will probably pass in arithmetic and fail in algebra and geometry. Some of the A division will take papers in advanced mathematics.

In Form I we usually devote one period a week to geometry and four to arithmetic for the first half year. After the half year we take two of the arithmetic periods for algebra. In Forms II, III, IV, V we devote two periods and one home work to arithmetic or numerical trigonometry; but any teacher has liberty to divide up the time as she chooses, provided the pupils are making satisfactory progress. All written work is done neatly in ink in exercise books both in class and at home. The only pencil work is usually trying rough figures for originals in geometry and also for diagrams. All work must be shown in the neat books.

Pupils entering the school will have learnt arithmetic as far as easy decimal and vulgar fractions. Much of the time in the first year is spent in drill on vulgar and decimal fractions as progress in other branches of mathematics and science is hindered if number work is poor. Decimals are taught from the metre ruler and pupils in Form I almost immediately do exercises in weighing and measuring in the Physical Laboratory, where they always use the metric system of weights and measures. Throughout the

school a variety of problems is taken in arithmetic, including percentages, areas, speeds, stocks and shares. In the examinations pupils are allowed to use logarithmic tables, algebraic symbols and graphical solutions. In a good division arithmetic is dropped in Form III or IV and numerical trigonometry is substituted for it. Some universities set questions in the geometry papers to be solved by numerical trigonometry, and others make part of the arithmetic alternative to questions in trigonometry in the school leaving examinations.

In beginning algebra the use of symbols and processes is only introduced as required in the solution of problems. Only sufficient drill to enable pupils to manipulate the symbols that are used in their problems is given. Simple processes are taught in the sequence required for solving simple equations. Graphs are used extensively throughout the course. First we take graphs of statistics and of simple functions not including negative quantities. Gradually the idea of a graph of a function is introduced and then the graphs may be used to solve equations of any degree. Examples are usually chosen of equation which are not more easily solved by simple algebraical methods. e.g. The graphic method of solving a quadratic equation which does not readily factorize is learnt before the method of completing the square or using the formula. In all graphical work the idea of variation is kept in mind. Work on maxima and minima and the gradient of a graph lead up to easy differentiation and work on areas to easy integration.

Only factoring of quadratic expressions is taken in preparation for work on quadratic equations and easy fractions. Logarithms are introduced in about the third year of algebra, and here again the graph of $10x$ between the values $x = 0$ and $x = 10$ gives a simple introduction to the use of logarithms before pupils are initiated into the use of the tables. Work on progressions follows the work on logarithms, so examples on geometric progressions can always be solved with the use of logarithms.

Harder factors and work on binomial and exponential and logarithmic series is only taken by pupils taking advanced papers.

In geometry we have a year of practical work. Children learn the use of ruler, protractor, compasses, and set-square. They draw accurately and measure many figures. They construct simple solids. They draw to scale and solve problems on heights

and distances. As a result of this year's work they acquire facility in using instruments and drawing neat and accurate diagrams, they become familiar with many geometrical terms, they acquire a working knowledge of many propositions on congruency, parallels, similarity, etc. Not least, they learn how to read a question and pick out the data and what is to be found. Very simple deductive exercises can be given on calculating the values of angles and the dimensions of similar figures.

In the next three years the pupils work through the usual course of propositions on plane geometry. We formulate no postulates or axioms, but in doing a deductive proof they are used without ever having been defined. Freedom of sequence is allowed in English schools. With beginners we formulate but do not give rigid proofs of fundamental theorems, such as those on congruency prove by superposition and the proof of two lines being parallel when alternate angles are equal. Statements of such fundamental theorems are however utilized as the basis of simple exercises which give the first introduction to a method of a formal deductive proof. The proofs as written out alternate between "since" and "therefore," but the pupil never considers it necessary to offer "identity" or "by substitution" in support of a statement. Practical work is never dropped and new propositions are often formulated as a result of drawing and measuring figures, before the proof is discovered. Definitions are not usually given until pupils are using the terms and can formulate them for themselves.

We take the propositions on proportion and similarity early in the course, after congruence and parallelograms, so that the pupils can easily become acquainted with the fact that trigonometrical functions of a given angle are constant, whatever size the figure. In introducing trigonometrical functions we use the graphs of the functions from 0° to 90° before introducing the tables. Graphs are constructed by pupils from their own accurate figures.

The actual method of conducting a class is left to the discretion of the teacher. The recitation method is not very common with us. Perhaps the most typical way with us is for the teacher to develop new matter by acting more or less as chairman to a class discussion, assisting by hints, questions and suggestions and keeping the class from digressing too much and

actually imparting information when necessary. Considerable practice with written work is given in class and the teacher then helps slow or backward pupils at her discretion.

The fifth year is used to review the whole field of plane geometry. Review work is often taken by working backward through all the propositions which are quoted in order to prove a particular question, and hence arriving at those particular fundamental theorems which were originally assumed. The pupil is now sufficiently advanced to appreciate a formal proof and even the formulation of axioms as the basis of her work.

Pupils in upper divisions may now do some solid geometry, conic sections or calculus.

If any mathematics is done in the sixth form, it is of an advanced type.

SOME LOVERS OF THE CONIC SECTIONS

MARGARET L. CHAPIN

The conic sections have been among the most beloved playthings of mathematicians for more than two thousand years. I remember that I was fascinated as a very little girl when my mother showed me a cone that had been sliced this way and that to give different curves; and I wondered whether I should ever study about them when I came to college.

No one knows who first thought of cones or of conic sections. Probably some old Greek geometer in the time of Pythagoras one day tried spinning a triangle around one of its sides to see what would happen. Cones seem to have been in good and regular standing at the end of the fifth century; for Archytas of Tarentum (cir. 400 B. C.) employs them in his solution of the cube-duplication problem, and a few years later Eudoxus gives us the formula for the volume of a cone and uses cones in his method of exhaustions.

Just who first sliced a cone we do not know. Probably ovals had been known for a good while; Euclid says that an oblique section of a cylinder gives a curve "like a shield." We first hear of the "Menaechmian triads" in the early 4th century B. C., because Menaechmus employs conics to solve the cube-reduplication problem. He needed only two conics to solve the problem, so very likely he or someone else had already discovered the curves in another way.

In the next fifty years, the theory of conics was rapidly developed. The method universally employed was as follows: Taking a right-angled triangle ABC, rotate it about one leg, say AB; obtaining a right circular cone. Pass a plane through the cone parallel to the line AC at some position in its round trips, obtaining a conic. This conic will be an ellipse, a parabola, or a hyperbola, according as the cone is "acute" or "right" or "obtuse angled"—that is, according as the angle at A in the triangle is less than, equal to, or greater than 45°.

It seems most unlikely that the keen mathematical minds of Alexandria never thought of cutting a cone in any other way;

but they probably preferred the above method as much more elegant. Both Euclid and Archimedes worked with conics from this point of view.

The work of Apollonius is of far greater generality. Taking a circle, he rotates a line in such a way that it continually follows the circumference and passes through a fixed point not in the plane of the circle nor in general on the perpendicular to that plane through its center. He obtains thus an *oblique* cone, and moreover a *double* one;—his predecessors, by simply rotating a triangle, had failed to obtain the second nappe of the cone, which had led to a neglect of the second branch of the hyperbola. He then proceeds to cut the cone by various planes, and shows that he obtains the ellipse when the plane cuts all the lines on the surface of the cone, the parabola when it is parellel to one of them, and the hyperbola when it is parallel to two of them. The methods of proportion which he employs to set up the properties of these conics are as elegant as any of the less general methods of his predecessors.

Apollonius wrote eight books on conics. Four of these contain propositions known and collected by Euclid, including the properties of axes, of asympoltes, of diameters; the discussion of intersections of various forms of conics; and the solution of a set of problems called the "solid loci" which seem to have been propounded by an earlier writer, Aristaeus: "If four lines are given, and from a point lines be drawn to meet the given lines at given angles, and if the product of the first two segments bears a constant ratio to the product of the other two segments, the point from which they radiate must have for its locus a conic"—the so-called four-line locus; and also the three-line locus, the case when two of the given lines coincide. The properties of the focus and directrix are set forth at length, and a great number of properties belonging to some one specific type of conic are proved.

The entirely original part of Apollonius' work, aside from his method of obtaining the sections from any general cone, is found in his later books, in which he discusses maxima, minima, limits, and normals—virtually a discussion of evolutes. He went about as far as it was possible to push the study employing the methods of pure geometry which were the Greeks' only tool; and for more than 1800 years his work remained the last word about conics.

The work of Apollonius had been completely generalized as regards the generation of conics; but thereafter he, and all the Greek geometers, had been interested to prove the particular metric properties belonging to one of the different types of conics, rather than those properties which belong to all conics. The first attempts to study the fundamental unity of these types seem to have been made in the sixteenth century A. D., but very little was accomplished until the time of Desargues.

The seventeenth century witnessed great developments in mathematical methods, and their application to the sciences. Desargues, Fermat, Mersenne, Descartes, Pascal, and finally Leibniz, Huygens, and Newton made their contributions to mathematics and science within this period. The invention by Descartes of the analytic method, swiftly followed by the development of the calculus, furnished a powerful tool to supplement the methods of the Greeks, by reducing the metric properties of curves to algebraic equations. It was discovered that the general equation of second degree in two variables—$ax^2 + by^2 + 2hxy + 2x + 2gy + c = 0$—always represents a conic; and conversely; and from the algebraic equations of tangents, asymptotes, and axes, conics could be swiftly drawn without resorting to the use of cones. The Greeks had, it is true, been able to construct conics entirely by means of plane methods, since they had discovered that a conic is the locus of points whose distance from the focus bears a constant ratio to the distance to the directrix; but the method of construction derived from this property is necessarily point-by-point; so that the invention of analytic geometry made the construction of any particular conic very much simpler. However, comparatively few new properties of conics were discovered; the knowledge of the Greeks had been merely reduced to a more convenient form. The methods of synthetic geometry were needed to discover certain other properties; and Desargues was really the founder of synthetic geometry.

Girard Desargues was born in Lyons in 1593. His education must have included Euclid and Apollonius; but we know almost nothing of his life until he appeared in Paris as a young man, about thirty, already distinguishing himself in the field of applied mathematics. Richelieu, Louis XIII's great prime minister,

seems to have been interested in his work, and we know that he served as an engineer and master-mechanic at the siege of La Rochelle in 1629.

"Applied mathematics" implied a great deal less then than it would today; even so, Desargues was a wonderfully versatile man. His studies included the use of perspective for architectural drawings; cutting of stones for use in buildings; mechanics; and the construction of sun-dials. He is known to have designed some actual buildings, and at one time Lyons invited him to design some public edifice, in order apparently to have an example of the work of a famous native of the town.

Desargues, however, was no mere bread-and-butter mathematician; he was one of the circle of pioneers in mathematics and science which flourished at Paris, whose members discussed such subjects as the Copernican theory, and read one another extracts from their manuscripts. Descartes, although living in Holland at that time for religious and political reasons, kept in close touch with the coterie, and greatly admired the work of Desargues. The latter did comparatively little work after the death of Descartes.

One or two pamphlets on Perspective preceded Desargues' most considerable published work—"Le Brouillon Project"—the Rough-Draft; which, though it contains short appendices relating to sun-dials and stone-cutting, consists principally of a Treatise on Conics, which contained the principles of the new method. This appeared in 1639.

The Greek geometers had met with difficulties in dealing with limiting cases in which two lines or two planes become parallel to one another and thus do not meet. Desargues seems to have first published, if not originated, the notion that parallel lines meet at an infinitely distant point. This notion makes possible the generality of the methods he employs. He further thinks of a line as the circumference of a circle of infinite radius: for he says, if a line-segment be made to rotate about one extremity, it will describe a circle or a straight line according as the center is at a finite or an infinite distance; hence, we may regard the circle and the straight line "as different species of the same genus."

The "*Brouillon-Project*" is difficult to read, for a number of reasons. Desargues would have found it much easier to express

certain theorems if he had used algebraic notation; moreover, his individual style is sometimes obscure, and his terminology is very puzzling. This is only natural; for, since he originated his methods, he had of course to coin names for its peculiar features, and he did this with an imagination almost poetic in spots. He translates the terms ellipse, parabola, hyperbola—which are of course Greek words—literally into French—"*défaillement,*" "outrepassement,"—"égalation"; he speaks of foci as "burning points"—evidently he knew about Archimedes' use of the parabolic mirror. In his work with the theory of involution he had to invent every term he employed, and his terminology does not happen to be that in use today. The base of an involution-range is called a *tree-trunk;* the involution is the *tree;* the center, the *stump;* a point of the involution, a *knot;* a segment measured from the center, a *branch*. The terminology further expands to include coupled knots and mean and extreme knots and twin coupled twigs, until one wonders whether he knew himself what he meant.

The theory of involution of six points is the basis of the entire work. It is defined by means of the ratio of the segments; its treatment as a projectivity came much later. Desargues especially loved the "involution of four points"—that is, the limiting case where two of the six points have come into coincidence with two others and have become double points. This, of course, is the range of four harmonic points, and he proves many familiar properties of the harmonic range, and also of the harmonic pencil of which it is a section; and finally he proves the basic theorem of his work: a pencil in involution cuts any range whatever in an involution, that is, the involution property is projective.

His method of generating conics is precisely that of Apollonius; except that in generating the cone he recognizes the cylinder as a cone with vertex at infinity.

The theory of poles and polars was originated by Desargues, though it is often attributed to La Hire, who lived about fifty years later; and he extended it moreover to cover the theory of polar planes and points in space.

Desargues then proves the theorem that opposite pairs of sides of a complete quadrangle cut any transversal in an involution, by means of a theorem of Ptolemy on a triangle cut by a trans-

versal. From this follows the famous theorem that a quadrangle inscribed in a conic cuts a transversal in an involution of which its intersections with the conic are one pair of points. This is first proved for a circle. Now take any other section of the cone standing on this circle as a base, and let the entire figure be projected upon this section. The involution property is projective. Therefore the theorem holds for any conic at all. The theory of involution, and the property that any conic may be derived from any other by projection and section, furnish the keys to Desargues' wonderfully general treatment. This theorem seems to have been discovered independently by Fermat, but he made no use of it.

Desargues goes on to sketch out hastily properties of coplanar conics, foci, conics and lines in their plane; the construction of the parameter of a diameter; and so on. He mentions a method of constructing conics which shall be perfectly general and confined to one plane; but he seems not to have worked it out for the public to appreciate.

Desargues' works were so unfortunate as to attract a great deal of hostility. One of the pamphlets on Perspective is known only from the quotations embodied in another pamphlet which bitterly attacked it. Desargues himself seems to have resented peculiarly any work in his field which was inadequate. Once a Jesuit priest issued a work on Perspective adopting Desargues' method but containing serious mistakes; and he went so far as to have notices put up on certain billboards disclaiming any responsibility for these grave errors. Also he seems to have had a tiff with the powers that be, when he disagreed with one Beaugrand, secretary to His Majesty, who had proved by geometry that a body weighs less when near the earth's surface! Finally he was himself attacked by a certain Curabelle in such a way that he laid a heavy wager on the correctness of his methods. When, however, he insisted that mathematicians only would be competent to judge, Curabelle complained that the jury would all be prejudiced in Desargues' favor, and proceeded to bring a lawsuit. Desargues thereafter published no more works but kept his discoveries to himself and to his pupils.

One of his pupils, Abraham Bosse, who became quite noted as an engraver, published some of Desargues' discoveries in his own works on Perspective; from these, among other things, we learn

the famous theorem about two perspective triangles. Bosse became a professor at the Academy of Fine Arts; and even there his connection with Desargues was brought against him, until he resigned his professorship rather than stop teaching his master's theories.

Desargues had retired to Lyons by 1650. The last years of his life he spent cultivating his garden and training a select company of stone-masons in his beloved technique of stone-cutting. He died in 1662, practically forgotten.

Desartes and Desargues were contemporaries, friends, pioneers. Each invented an important method. Curiously, the method of Descartes came to immediate fruition, while that of Desargues lay untouched after his death for almost 150 years. This is probably explained in part by the enmity which Desargues was so unfortunate as to excite, and partly by the limited circulation and comparative difficulty of his works.

One at least of Desargues' contemporaries appreciated his work, and lent the glamor of a brilliant name to keep him from sinking into entire oblivion. At the meetings of the mathematical club above referred to—which was the forerunner of the Academy of Sciences—Descartes had probably met Étienne Pascal, who had held various positions under the government, and is better known to us because of his investigation of certain plane curves. He had at this time three remarkable children—two daughters and a son.

The story of Blaise and Jacqueline Pascal, the younger of these three, as told long after by their older sister, Madame Périer, is to me very pathetic. Both were remarkably precocious children. The father, wishing to give the boy a thorough training in the classics, refrained from teaching him algebra and geometry; but Blaise had so strong a natural aptitude that he studied them secretly and amazed his father by his unaided attainments. Jacqueline of course was not taught these subjects; but she had a strong taste for poetry, and before she was fairly in her teens she had been introduced at court and made to display her skill in making verses on the spur of the moment. Her poems are quaint specimens of the artificial 17th century style, and are chiefly on religious themes or addresses to persons at court.

The brother and sister, grown to young manhood and womanhood, occupied a conspicuous social position in Paris, and are said both to have possessed very striking beauty. Blaise continued to work along mathematical and scientific lines—at nineteen he devised and patented the first known computing machine, and a few years later he and Roberval were experimenting with vacua and measuring atmospheric pressure on the Puy de Dome—but the family seems to have held a position in the social as well as the intellectual life of the day. By degrees, however, a latent religious bent revealed itself in both brother and sister, and they were both attracted to the Jansenists—the stricter and more Puritanical of the French Catholics. Both finally entered the convent of Port Royal; and the brilliant scientist and writer and his lovely sister devoted their lives thenceforth to religious meditation and the writing of devotional literature.

Pascal states himself that his interest in conics is largely due to Desargues, whom he mentions as "one of the greatest minds of this age." Probably Pascal had met him as a boy of fourteen—at which time, by his sister's account, he had been studying mathematics only a year or two. The "Brouillon-Project" appeared in 1639, when Pascal was barely fifteen; and undoubedly he read it through enthusiastically. In 1640—the following year—Pascal himself wrote a modest little essay which was printed at Paris; and the contents were recognized as so remarkable that no one would believe a boy of sixteen was the author. Descartès openly said the father must be passing off his own work in this way. Pascal began to write a much fuller treatise on conics, but this was never completed, and practically all the fragments are now lost.

The Essay on Conics of 1640 borrows directly from Desargues the method of generating the conics and the notions regarding the meeting of parallel lines. The famous involution-theorem is quoted, with mention of the authority. The rest of the paper is composed of the enunciation without proof of certain theorems which are given by Pappus, which—Pascal claims—follow at once from the theorem that has ever since borne his name: opposite sides of a hexagon inscribed in a conic intersect in three collinear points. Apparently he first proves this theorem for a circle, and then by projection demonstrates its truth for all types of conics. The basis of Pascal's never-completed work then

seems to have been the perspective relationship existing between types of conics, and his hexagon theorem.

The brilliant boy must have been petted and made much of by the Paris mathematicians. The hexagon whose opposite sides meet in three collinear points was soon known as the "mystic hexagram." Pascal had modestly referred to his theorem as having "a great many corollaries" which he would state when God should give him strength, and some of these he must have communicated to the Paris mathematical circle, for Mersenne wrote of him in 1644, "Unica propositione universalissima, 400 corollariis armata, integrum Apollonium complexus est"—in one most general proposition with 400 corollaries he has put all of Apollonius.

We should very much like to see just how Pascal derived four hundred corollaries from his proposition. More recent writers have discovered remarkable properties of a Pascal hexagon by considering the relations existing between the Pascal lines of the 60 different hexagons obtained by joining the vertices in different orders, but Pascal could hardly have obtained all the results of Apollonius by means of these. The exhaustive treatise which never was published would probably answer this question. Part of the work was at least in the form of a rough draft for Leibniz and Mersenne both mention having seen a part of it. Mersenne, who evidently ran to flowery language, wrote to a friend of Huygeus. "If your Archimedes comes here, we will show him what our young Apollonius has been doing." We have a better source of information in a letter which Leibniz wrote to Pascal's nephew some ten years after his death. Leibniz had borrowed all available papers, and was urging that they be published; he speaks of the colored figures which go with the manuscript. The contents seem to have been a discussion of poles and polars; of ratios between secants and tangents; of the problem: to draw a conic tangent to five given lines, no three concurrent; of the "three-line" and "four-line" problems which Apollonius had worked with; of the construction of a conic given a diameter and a parameter; and of the problem: given a cone to cut a section similar to a given conic. These works were never published, and are probably lost; with one exception.

A fragment of a few pages, evidently the introduction to the entire work, entitled *Generatio Conisectorum*, has been found.

It contains rather little original matter, but is a careful discussion of the three types of conics as projections of a circle. Pascal has entirely adopted the notion that parallel lines meet at an infinitely distant point; but he prefers to say that two points on the hyperbola, for instance, are "lacking." This treatise was written in 1648—only eight years later than the Essay on Conics; but its carefully exhaustive style is in striking contrast to the sketchy exuberance of the subject matter in the Essay.

Pascal did far less than Desargues, but his name was much longer remembered, largely because he had made a brief and phenomenal career and had retired from the scene before people had had a chance to weary of him.

The development of modern synthetic geometry began to take place about 1790. The works of Desargues were rediscovered and made the basis of the work of Poncelet and later of Chasles, both of whom endeavored to give him due credit. It is curious that the last general method of generating conics to be discovered is from most points of view the simplest: the conic as determined by the intersections of pairs of corresponding rays of two projective pencils. This method is perfectly general, and is confined to the plane of the conic.

In making these discoveries mathematicians of course unearthed many fields for further study; so that conics can no longer be regarded as the latest thing in curves; and those who would cover the field of even plane analytics must discuss conics with a comparatively cursory survey today. But they will always continue not only important as starting-points in analytical and synthetic geometry but beautiful for their underlying unity and for the wealth of historical associations which have gathered about them.

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SOME HIGHER ASPECTS OF SECONDARY SCHOOL MATHEMATICS

An address delivered before the Mathematics Section of the Alabama Education Association, April, 1925.

By CHESTER G. BANDMAN
Principal Woodlawn High School, Birmingham, Ala.

It is a significant and compelling fact that an inordinate percentage of high school pupils regard the subject of mathematics as drudgery and that the percentage of failures is usually larger in mathematics than in almost any high school study. There is undoubtedly a close correlation between the two statements.

The question naturally arises: What are we giving our students of mathematics that is of permanent value? If we are failing to inspire them and, as it seems to them, conspiring to fail them, what are we accomplishing? It would be interesting and informative to submit a questionnaire to high school graduates, inquiring of them what they had gained from their study of mathematics. Unquestionably most of them would be at a loss to assign a single constructive benefit they had derived. And why? Because we have failed to recognize and take advantage of the magnificent, the practically boundless opportunities presented by this majestic study to impress upon the pupils some of the noblest thoughts, some of the greatest virtues known to men.

All of you are more or less familiar with the U. S. Bureau of Education's bulletin entitled "Cardinal Principles of Secondary Education," in which are set out the seven objectives of education, namely, (1) health, (2) command of the fundamental processes, (3) vocation, (4) worthy home membership, (5) citizenship, (6) use of leisure time, and (7) ethical character. It does not come within the province of this paper to discuss these various objectives of education, but it is most pertinent to raise the question: How and to what extent does the study of algebra, geometry and trigonometry justify itself in the fulfillment of these objectives? Unless these high school branches function in at least one of these seven respects, they are not worthy to be included in the curriculum. Do they, then, contribute to the promotion of one or more of the above-mentioned objectives?

It shall be the purpose of this discussion to point out in a very meagre way how the dry-as-a-bone subject of mathematics may be used in the development of ethical character. It is axiomatic, of course, that, in order to aid in that development, the subject must be presented by a teacher thoroughly imbued with the sublimity and the beauty of mathematics. Many years ago I came across a bit of verse which characterizes too many members of our profession; I trust it isn't typical of the group of mathematics teachers assembled here today. It reads as follows:

A PURE MATHEMATICIAN

Let poets chant of clouds and things
In lonely attics,
A nobler lot is his who clings
To Mathematics.

Sublime he sits, no worldly strife
His bosom vexes,
Reducing all the doubts of life
To Y's and X's.

And naught to him's a primrose on
The river's border,
A parallelopipedon
Is more in order.

Let braggarts vow to do and dare
And right abuses;
He'd rather sit at home and square
Hypotenuses.

Along his straight ruled path he goes
Contented with 'em,
The only rhythm that he knows—
A logarithm.

The teacher who sees in mathematics only a medium of imparting an understanding of the manipulation of abstract letters and figures, who places emphasis only upon the mechanics of the subject, is overlooking an opportunity too vast for words; such a teacher is sadly lacking in vision. Small wonder that pupils of such a teacher find little inspiration in pursuing the study of mathematics; small wonder that they develop a loathing for it and welcome the day when they shall have finished all the mathematics required and may choose something more inspiring, more fascinating, more elevating!

In September, 1921, Dr. David Eugene Smith, Professor of Mathematics, Teachers College, Columbia, delivered an address

before the Mathematical Association of America, entitled "Religio Mathematici," in which he pointed out marks of similarity between mathematics and religion. So broad, so all-inclusive does he find the realm of mathematics that he makes bold to assert that many of the truths usually associated with religion only may be equally strongly impressed by means of the more prosaic principles of algebra and geometry. We ordinarily confine our thoughts on the subject of immortality to theological disputations; we think of metaphysics as belonging exclusively to the philosopher and the clergy. And yet, Dr. Smith demonstrates that immortality may well be brought out in a beginners' algebra class, provided the teacher is skilled and knows how to handle tactfully a situation that might sometimes prove difficult and even dangerous. I quote: "One thing that mathematics early imparts, unless hindered from so doing, is the idea that here, at last, is an immortality that is seemingly tangible,—the immortality of a mathematical law. The student of algebra, for example, may well question the use of the traditional curriculum, but when he finds the value of $(a + b)^2$ he has come in contact with an eternal law. The laws of the Medes and Persians, unchangeable though they were thought to be, have all perished; the canons that bound Egyptian activities for thousands of years exist only in the ancient records, preserved in our museums of antiquity; the laws of Rome, which at one time dominated the legal world, have given place to modern codes; and the laws that we make today are certain to be changed tomorrow. But in the midst of all these changes it has ever been true, it is true today, it shall be true in all the future of this earth, and it is equally true throughout the universe, whether in the algebra of Flatland or in that of the space in which we live, that $(a + b)^2 = a^2 + 2ab + b^2$. We may change the symbols,—they are the temporary expedients to convey the idea; we may speak in different tongues,—they are local expedients to convey thought, but it is inconceivable to us that the relation which the formula expresses should not be true always and everywhere,—a tangible symbol of the immortality of law."

Here, then, is a concept which the mathematics teacher with vision may well pause to convey to the high school student; here is an opportunity to open up a channel of thought which might otherwise be dammed up forever. Such fundamental ideas, intro-

duced through the agency of algebraic formulas, geometrical theorems, trigonometric principles, will not only bring the adolescent child to a realization of the eternal verities, they will not only convince him of the ethical values bound up in elementary mathematics; but they will give him an urge, a stimulus to continue his study; they will arouse his interest and his imagination; they will cause him to look back in later years upon his high school mathematics as the instrument which widened his horizon and broadened his life.

Often have I had pupils look at me in open-eyed amazement, and regard me with skeptical distrust when I have told them truly that a perfect proof of the Pythagorean proposition is more beautiful to me than the immortal picture of Raphael's Sistine Madonna which hangs in the Art Gallery at Dresden. Of course, the reason why that is true is that my mathematical training has given me the power to appreciate the beauty of the proof that the square on the hypotenuse of a right triangle is equivalent to the sum of the squares on the two sides; while my training in appreciation of art has been inadequate. The point at issue is that cold reasoning and the technique of the formal demonstration of a mathematical proposition are by no means the be-all and the end-all of an hour's instruction in secondary school mathematics. Truth itself is a worthy objective, but it may be worthily attended by its handmaiden, the beauty of truth. Although the author of the following poem, who, by the way, is none other than our own Arthur F. Harmon, Superintendent of Education of Montgomery County, did not have in mind any specific subject, the poem can easily be applied to the topic in hand. As I read it, see how aptly it may be related to mathematics:

'TIS THIS TO TEACH

To take a child in gentle hands
    And lead him into mystic lands,
Where veils no longer shroud the past
    And each new hope o'erglows the last—
'Tis this to teach.

To light new fires where old have burned,
    With brave, good hearts, as roads are turned,
To find new stars where darkness sways,
    Whose light shall one day mark the ways—
'Tis this to teach.

To fill the child world brim with joy,
    To charm and hold some errant boy
With stern ambition, or some song
    Of right triumphant over wrong—
'Tis this to teach.

To move dread mountains dark with fear,
    By faith of young hearts drawing near
The paths the fathers long have trod,
    The narrow paths that lead to God—
'Tis this to teach.

Within the limited confines of this talk, it is impossible even to cite the innumerable ethical aspects which mathematics lends itself to in the hands of a teacher thoroughly imbued with the spirit of his subject. I might point briefly to a few of the contacts which are treated so beautifully and convincingly in the treatise by Dr. Smith, to which reference was made earlier in this paper. Prof. Smith shows that, in addition to the lesson of immortality, pure mathematics may be used to demonstrate to the growing boy and girl our infinitesimal nature, our contact with the infinite, our impotence in relation to the eternal, and the permanence of physical things. I know of no better way to close this discussion than to quote the conclusion which Dr. Smith reaches in his masterful essay: "And what is the conclusion? Does mathematics make a man religious? Does it give him a basis for ethics? Will the individual love his fellow man more certainly because of the square on the hypotenuse? Such questions are trivial; they are food for the youthful paragrapher. Mathematics makes no such claim. What we may safely assert, however, is this,—that mathematics increases the faith of a man who has faith; that it shows him his finite nature with respect to the Infinite; that it puts him in touch with immortality in the form of mathematical laws that are eternal; and that it shows him the futility of setting up his childish arrogance of disbelief in that which he cannot see.

"And if this be the case, then what is the duty of teachers of our science? To preach?—that should be the last thought. The greatest sermons are preached in silence. The most ancient religions that we have, if there be more than one fundamental religion, have always recognized this fact. And so it must be with us,—that we should teach "the science venerable" not merely for its technique; not solely for this little group of laws or that; not only for a body of unrelated propositions or for

some examination set by the schools; but that we should teach it primarily for the beauty of the discipline, for "the music of the spheres," and for the faith that it gives in truth, in eternal law, in the Infinite, and in the reality of the imaginary; and for the feeling of humility that results from our comparison of the laws within our reach and those which obtain in the transfinite domain. With such a spirit to guide us, what teachers we would be!—whether of those who are standing on the threshold, of those who are passing through the realms of mystery that lead to manhood and womanhood, of those of mature years, or of those who, as the ancients were wont to say, 'number their years upon their right hand.' "

Let us, then, fellow teachers, resolve to go back to our respective tasks with the firm determination to bring to our boys and girls a vision of the higher aspects of our beloved subject, let us put flesh and blood on the dry bones and strive to put life into a dead body. In a word, let us grasp the nobility, the grandeur, the sublimity of the king of sciences, and, in the spirit of the true teacher, endeavor to endow our pupils with a double portion of our inheritance.

NEW BOOKS

Plane Geometry by F. Eugene Seymour. American Book Company. Pp. 333, 1925.

Methods of teaching slowly but surely reflect the prevailing psychology of the day. In no subject, however, has the influence of modern psychology been slower in effecting teaching than in plane geometry. Our text books present complete proofs of the great basal propositions of geometry. We teach the texts as they are written. To study these propositions the pupil *reads,* and *rereads* their proofs until he can reproduce them. The pupil, of course, does not discover, is not challenged, does little thinking, and hence certainly gains little in power in thinking, the development of which is the chief aim of teaching plane geometry.

Mr. Seymour, as Supervisor of Mathematics in the New York State Department, has had rare opportunity to observe the stultifying effects of this memoriter work in the treatment of the basal propositions. To help point the way to a better day in the teaching of plane geometry, he has written a text of such nature as to require the pupil to "study" the propositions, rather than to "read" them.

"Throughout this text, except in those instances where the proposition is so simple that the proof is left entirely to the pupil, the author attempts, by an informal analysis, consisting of a series of natural and logical questions, to get the pupil to see how one would go about to discover the proof from the material immediately given and already at hand." . . .

Complete formal proofs appear in sufficient numbers to serve as models. Careful attention has been given to the systematization and organization of methods of proving originals, which occur in great abundance.

The reviewer believes that Mr. Seymour's text, if used as written, will be considerably more effective than a similar use of the old type of text, both in arousing pupils' interest in geometry and in increasing their skill as real discoverers of geometrical truths and proofs.

Cumulative Mathematics by D. W. Werremeyer. Harcourt, Brace and Company, New York.

Cumulative Mathematics is a three-book series, planned for the seventh, the eighth and the ninth years, respectively.

As the title implies, the author has used what he is pleased to call the *cumulative method* of presentation. By means of this method, the habit of forgetting on the part of the pupil is reduced to a minimum in that the pupil is constantly required to use repeatedly every new principle he has learned in all subsequent problem material. Not only is the problem material of each book cumulative as regards the subject matter of that book, but the material of the eighth year book and of the ninth year book is cumulative as regards all antecedent material. This particular feature of this series should make a strong appeal to teachers as it has long been the experience of all teachers that the pupils have forgotten much of the material which they have previously learned.

The author appreciates the need for drill in the four fundamental operations with both integral and fractional numbers and has so organized the drill material that the work may be done for several consecutive weeks or distributed throughout the entire term for five or ten minutes daily.

Motivation is taken care of by means of introductory concrete problems which show the need for each process taught.

At suitable intervals throughout the series, mastery tests, cumulative reviews, and contest exercises have been inserted, the object of which is to secure genuine skill in the essential operations. Every effort has been made to convince the pupil that no computation is complete until he has checked it.

Moreover, the author talks to the pupil in language so clear, concise, and succinct that little, if any, help on the part of the teacher is required. This series of books has been written *to* the pupil and *for* the pupil.

The material in the entire series is planned so as to take care of groups of different abilities. The slower groups may omit the starred topics and some of the longer lists of cumulative problems which are graded according to difficulty. The average groups should just cover the text; and the accelerated groups will find sufficient additional work in the supplements.

The Seventh Year Book begins with "An Exercise in Following Directions." Such an exercise as this is sorely needed for the weakness of pupils in this respect is proverbial and the cause of much of the poor work which is done.

Work on graphical representation of facts comes early in the book. The pupil learns not only to make graphical records of his grades, of games lost and won, of daily temperatures, etc.; but also to interpret graphs which he finds in the newspapers and magazines.

The material offered under "Problems in Everyday Life" emphasizes the *social* uses of mathematics and associates the school work closely with the life of the pupil outside the classroom.

The equation method of solving problems is introduced early in the seventh grade. The equation principle of division is the only one used, because it is the only one needed in this year. It has been the experience of many teachers that the pupil likes to use the equation method because it is a shorter method of solving problems.

Percentage is not taught by cases. When the pupil has been shown that the per cent sign is used merely to write a fraction in another form; that is, to express a fraction as hundredths, then it is clear to him that he may solve all problems in percentage in exactly the same manner as those involving common and decimal fractions.

The material on business arithmetic comes within the experience of the child. The problem method of introducing a new topic has been generously used so as to challenge the interest and arouse the reasoning faculty of the pupil. The interrelations between profit and loss, commercial discount and commission have been stressed by means of cumulative problems.

The subject of interest grows out of the rent idea. This discussion is followed by some of the simpler and practical phases of banking. By means of an additional list of cumulative problems, the various phases of business arithmetic have been closely interwoven so as to approximate life situations.

The Eighth-Year Book deals primarily with mensurational geometry, community arithmetic, arithmetic of investments, the fundamental operations with *positive* literal numbers, and the

simple equation. The aim of the work in the mensurational geometry is to direct the pupil to do his own observing, describing and thinking. Great pains have been taken to develop the topics by the heuristic method. By this method, the pupil discovers for himself the characteristics of lines, surfaces and solids. The classroom, the home and the pupil's surroundings become laboratories for this work. Much emphasis is placed upon scale drawings because, worthwhile in themselves, they also involve an unconscious review of arithmetic.

Realizing that complete comprehension is the first step toward mastery, all formulas are carefully rationalized, so that the pupil may understand the basis for the reasoning in each case. More than ordinary care is taken to imprint indelibly upon the pupil's mind the best methods of substituting in formulas, and solving for any letter in the formula. Algebra is introduced only as the need for it arises and in such a way as to identify it as a part of mathematics.

The work on community arithmetic and arithmetic of investments has been postponed until the latter part of the eighth year, because experience has shown that the pupil can live these situations better at this age. The subject of taxes is introduced by studying how the expenses of an individual and of a club are met—thus to work from the pupil's own experience to the less familiar problems of the city, the state, and the Federal Government.

The work on insurance is limited to the most common-form policies and is simplified so as to come within the understanding of the average pupil.

Likewise, to show the necessity of selling stocks and bonds, there is a careful treatment of business development indicating how business develops from that of an individual owner to that of partnership, and finally to a corporation.

The material in the Ninth Year Book is mostly algebraic, but it is based upon the arithmetic and the mensurational geometry, which the pupil has had in the seventh and the eighth years.

The material on signed numbers is presented so that the pupil will always have practice on a new operation with positive numbers first, followed immediately by negative numbers; that is, only one new difficulty is presented at a time.

The sequence of the four fundamental operations differs from the traditional order in that multiplication follows addition; subtraction follows multiplication; and division follows subtraction. By means of this sequence, addition may be immediately applied in multiplication and subtraction, in division.

Each equation principle or axiom used is illustrated both geometrically and arithmetically in the simplest way. The pupil is then ready not only for the abstract equation but also for word problems involving the use of the equation. The material on the use of the simple equation is followed immediately by the material on the use of simultaneous linear equations. This arrangement lessens the abstract work during the first half of the ninth year and gives more practice in the solution of word problems.

The second half of this year's work begins with special products and factoring; making use of only those types which have been recommended by the National Committee on Mathematical Requirements. These types of factoring are put into practice immediately in the solution of the quadratic equation by factoring; in the reduction to lowest terms; in multiplication and division of fractions; and in addition and subtraction of fractions.

This arrangement of the material on special products, factoring and fractions not only reduces the amount of time usually spent on these topics but also makes it possible to teach it more effectively as the pupil will have the same teacher while taking these related topics.

The material in quadratic surds is in accord with the recommendations of the National Committee on Mathematical Requirements. The roots of numerical expressions have received the most stress and are immediately applied in the material which the pupils have had in mensurational geometry.

The material on certain topics has been placed in the supplement to furnish additional material for classes which have time to do it. The work on numerical trigonometry which was recommended by the National Committee on Mathematical Requirements is placed in the supplement so that schools which prefer to take this material in the ninth year may do so.

GEORGE P. KERR,
Cleveland, Ohio.

Uberlick uber die Geschichte der Elementarmathematik, by Dr. W. Lietzmann, B. G. Teubner, Leipzig and Berlin. 68 pages, 1926.

This little work, which has just come from the press, is written by Professor Lietzmann, whose name is well-known in this country because of his contributions to the history and teaching of mathematics. As its name indicates, it is a kind of bird's-eye view of the development of mathematics as a whole and is intended to condense in the short space of sixty-five pages a résumé of the chief steps in the progress of the science.

There is first a general survey of the development of elementary mathematics. This is followed by eight chapters on the subjects specially considered. These are, first, computation; second, the operations of algebra, which go in Germany by the name Arithmetik; third, algebra, which, as usual in German, refers to the equation; fourth, planimetry, covering the question of the finding of areas and the making of constructions, in addition to the development of the theoretical parts of geometry; fifth, stereometry; sixth, trigonometry; seventh, analysis, which includes analytic geometry, the concept of limits, and a brief statement concerning infinitesimal calculus, and a page on infinite series; eighth, conic sections, including the synthetic, analytic, and perspective geometry of conics.

At the end of the work there is given a list of the leading names of workers in the development of mathematics, together with their dates. This furnishes a convenient reference table. Just why Leonardo Fibonacci should have been given the dates *c.* 1200-1250? is, however, a question.

The pamphlet contains a considerable number of illustrations, most of them quite satisfactory. They naturally run into the German field rather than into the international one. The best portrait of the lot is that of Newton, which is taken from the well-known painting in the National Gallery. The poorest is that of Descartes.

As a small book of reference it will be found of value.

David Eugene Smith.

Der Gegenstand der Mathematik im Lichte Ihrer Entwicklung, by Dr. Heinrich Wieleitner. B. G. Teubner, Leipzig, 1925. Price 1 gold mark. 61 pages.

This little work of only 61 pages is an illustration of the best type of popular brochures on scientific subject appearing in Germany in recent years. It is one of the series "Mathematische-Physicalische Bibliothek," edited by Professors Lietzmann and Witting. It gives in very brief form the development of our modern mathematics through the ages. The work is divided into six chapters as follows:

I. General notes on mathematics and its development. II. The geometry of the Greeks, including elementary geometry, conic sections, and the introduction of the infinitesimal. III. Algebra. IV. Modern geometry. V. Higher analysis. VI. Mathematics and reality (Wirklichkeit).

The nearest parallel that we have to this in our language is perhaps Jourdain's work on "The Nature of Mathematics." Teachers who read German and who wish a brief synopsis of the development of mathematics would do well to send for a copy of this interesting brochure.

David Eugene Smith.

A TENTATIVE PROGRAM FOR THE ANNUAL MEETING OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

The next meeting will be at the Raleigh Hotel, Washington, D. C., Saturday, February 20th, 1926. The first program will begin at 10:00 A. M. The afternoon program begins at 2:00, and the dinner is at 6:00.

No doubt members of the Council know that the annual meeting of the Superintendence Section of the National Education Association begins on the day following our meeting. Aside from offering excellent programs to the members of the Council, there is the opportunity to obtain railroad fare to Washington and return at reduced rates,—one and one-half fare. There are at least four reasons why members of the Council should attend the meeting: (1) participation in the program,—the best the officers could collect by wide solicitation for materials; (2) the opportunity to attend meetings of the Superintendence Section of the National Education Association; (3) the chance to see Washington; (4) the probability of renewing old acquaintances.

The program is built around the theme "A General Survey of the Progress of Mathematics in our High Schools in the Last Twenty-Five Years." The Yearbook Committee, of which Mr. Charles M. Austin is chairman, will present a Yearbook, as directed by action taken at the Cincinnati meeting.

The complete program can not be given at this time because not all the material for the yearbook has been received. The general character, however, is indicated by the following:

1. Presentation of the Yearbook—Charles M. Austin, Chairman.
2. Paper on the general theme—Professor David Eugene Smith, Teachers College.
3. A review of Professor Moore's Presidential Address (reprinted in the Yearbook)—Mr. Harry English, Washington, D. C.
4. The Development of Mathematics for the Junior High School—Mr. William Betz, Rochester, New York.

5. "The Present Status of the Testing Movement as Concerns High School Mathematics"—Professor William D. Reeve, Teachers College.
6. "Recent Development in Mathematics Clubs"—Miss Marie Gugle, Columbus, Ohio.
7. "Mathematics Materials Published for our Schools in Recent Years"—Edwin W. Schreiber, Maywood, Illinois.
8. "Recent Investigations on the Teaching of Arithmetic"—Professor Frank Clapp, Madison, Wisconsin.
9. Presidential Address (as directed by the action at Cincinnati)—Professor Raleigh Schorling, University of Michigan.
10. Address on the general theme at the evening meeting by Supt. Frank Ballou, President of the Superintendence Section of the N. E. A.
11. Elective Courses in Senior High School Mathematics—Gordon R. Mirick and Vera Sanford, The Lincoln School.
12. Mathematics and Life—Professor H. E. Slaught, University of Chicago.
13. Orthodoxy and Heresy in Geometry Teaching—George W. Evans, Boston.

The Yearbook will be in the usual form of such documents and will contain at least 150 pages. It will sell for $1.10, including postage. At the Washington meeting, copies may be obtained for $1.00 per copy. Orders should be placed by writing directly to Charles M. Austin, Oak Park High School, Oak Park, Illinois. Those who expect to use the Yearbook in connection with teacher training courses should place their orders (through their local bookstores) with Mr. Austin at once for the reason that the officers of the Council not having a large reserve fund, could not venture the printing of a large edition. It is entirely possible that the edition will be exhausted very soon.

It is hoped that the program at Washington will appeal to teachers in the elementary schools, junior high school, and senior high school. For example, the address by Professor Clapp will be of very great interest to all teachers of arithmetic.

Raleigh Schorling,
President of the National Council of
Mathematics Teachers.

PROBABLE CONTENTS OF THE FIRST YEARBOOK ISSUED BY THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

General Theme: The Progress of the Past Twenty-five Years in the Teaching of Mathematics in Elementary and Secondary Schools.

1. President's Address—Raleigh Schorling.
2. Reprint of Presidential Address of E. H. Moore before American Mathematical Society.
3. Twenty-five Years of Progress in the Schools of England—T. Percy Nunn.
4. Story of the Progress of the Past Twenty-five Years in the United States.
   (a) The Syllabuses
   (b) Work of International Commission
   (c) Work of National Committee
   (d) Changes in Subject Matter
   —David Eugene Smith.
5. A Summary of the Testing Movement—W. D. Reeve.
6. By-products from the General Program in Education as Concerns the Teaching of Mathematics—E. L. Thorndike.
7. Mathematics of the Junior High School—William E. Betz.
8. A Critical Evaluation of the Things that Progress Has Brought Us—Harry English, F. C. Touton.
9. Mathematics Clubs in the High School.
10. Bibliography of Books Published Since 1920—Edwin W. Schreiber.
11. Elective Courses in Senior High School Mathematics—Gordon R. Mirick and Vera Sanford, The Lincoln School.
12. Mathematics and Life—Professor H. E. Slaught, University of Chicago.
13. Orthodoxy and Heresy in Geometry Teaching—George W. Evans, Boston.

This will make a book of 125 to 150 pages.

The book will be ready for distribution at the time of the annual meeting in Washington, D. C., February 20, 1926.

The price will be \$1.10.

All profits from the sale of the Yearbook go into the treasury of the Council. There are no fees to contributors or members of the committee. A complete statement of the costs of publication and distribution will be published as soon as possible.

This book is a project of the National Council and should have the support of every member of the organization. Every one who buys a book not only aids the Council but receives value received for his money.

The Committee hopes that a large advance subscription will follow this announcement.

Send all names and remittances to C. M. Austin, High School, Oak Park, Illinois.

HARRY ENGLISH,
WILLIAM E. BETZ,
W. C. EELLS,
F. C. TOUTON,
C. M. AUSTIN,
Yearbook Committee.